Existence of minimal models for varieties of log general type.

Introduction.

Goal

$\boxed{\text{Theorem 1}}$ Let (X, Δ) proj. klt.
Δ is big, $K_X + \Delta$ pseudo-effective.
$\Rightarrow$ $K_X + \Delta$ has a log-terminal model.

{ log terminal model = nef model = minimal model

{ log canonical model = ample model

many l.t.
unique l.c.

Theorem 2 If $K_X + \Delta$ big,
then $K_X + \Delta$ has a log canonical model.

$\boxed{T.1 \Rightarrow T.2}$

We can do $(1+\varepsilon)(K_X+\Delta)$

$$= K_X + \underbrace{\Big(\underbrace{\Delta}_{\text{eff}} + \underbrace{\varepsilon(K_X+\Delta)}_{\text{big}}\Big)}_{\text{big.}}$$

(ε is small enough).

We are in the conditions for $\boxed{T.1}$

$K_X+\Delta$ big and nef

base point free theorem, says

$\Rightarrow K_X+\Delta$ is semiample.

by taking $|mK_{X_{min}}|$

$$X_{min} \longrightarrow X_{can} \subset \mathbb{P}^M$$

Outlines of The MMP.

Origins

X smooth surfaces.

- If $\kappa(X, K_X) \geq 0$.

$\exists!$ Y bir. to X, such that

K_Y is nef.

Just gotten by contracting

[-1-curves. [C are rational
$C \cdot K_X = -1$.]]

- If $\kappa(X, K_X) = -\infty$ X is bir.

ruled surfaces.

[fibration over curve]

For higher dim.

- $\kappa(K_X) \geq 0$, get a min. model Y bir. to X, with $\underline{K_Y \text{ nef}}$.
- $\kappa(K_X) = -\infty$, Y bir. to X, s.t.

$\exists Z$, $\varphi: Y \longrightarrow Z$ with general fibre $\underline{\text{Fano}}$.

"$\underline{\text{Ex}}$" $\boxed{\text{no smooth min. model for some 3-folds.}}$

smooth compact 3-fold X.

$$\left[H^0(\mathcal{O}_X(mK_X)) \approx \frac{m^3}{4} \right].$$

$\underline{\text{Proof}}$) If X $\underline{\text{bir.}}$ to Y, K_Y is $\underline{\underline{\text{nef}}}$.

$H^0(\mathcal{O}(\ldots)) \ldots m^3$

$$H(\mathcal{O}_Y(mK_Y)) \approx \frac{\quad}{4}$$

mK_Y are big, and nef

So, we use Kawamata-Viehweg Vanishing:

$$H^i(\mathcal{O}_Y(mK_Y)) = 0$$

$$\boxed{\frac{m^3}{4}} \approx |H^0(\mathcal{O}_Y(mK_Y))| = \underbrace{\chi(\mathcal{O}_Y(mK_Y))}_{RR}$$

$$\frac{1}{4} = \boxed{\frac{(K_Y)^3}{3!} \cdot m^3}$$

$$\underline{\underline{(K_Y)^3}} = \frac{3}{\boxed{2}}.$$

If Y is smooth, then $(K_Y)^3$ is an integer.

- At least we want the K_X to be ~~$\mathbb{Q}$~~ Cartier.

$$H^0(mK_X) = H^0(mK_Y)$$

we'll want $X \dashrightarrow Y$ to be K_X-negative.

2 ways to get the MM.
("for the K_X pseudoeffective").

1st

"Using" canonical ring.

$$R(X, K_X) = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mK_X))$$

if it is f.g. and K_X is big.

$$\Rightarrow \mathrm{Proj}(R(X, K_X))$$

is the canonical model.

2^{nd} Make steps where K_X-neg. at each step

- If K_X not nef.

By the Cone theorem

$\Rightarrow \exists\, C \subset X$ rat. curve.

$K_X \cdot C < 0$.

and a morphism

$$f : X \to Z. \quad \text{surj.}$$

contracts irred curves D iff

$[D] \in \mathbb{R}^+[C]$ (up to numerical equivalence)

$\rho(X/Z) = 1$, and K_X is f-anti ample.

a) $\dim Z < \dim X$. this is Fano Fibration

Mori Fiber Space.

b) $\dim Z = \dim X$, f contracts a divisor.

divisorial contraction

c) $\dim Z = \dim X$, f does not contract divisors,

f is small contraction.

K_Z cannot be $\mathbb{Q}$-cartier

If it was, mK_X, mK_Z cartier.

mK_X, $f^*(mK_Z)$ are l.e. outside $Ex(f)$. as this has codim ≥ 2.

$\Rightarrow$ l.e. on X. so

$C \cdot mK_X < 0$

$\xi \cdot (f^*(mK_Z)) = 0. \quad (\Rightarrow\Leftarrow)$

$\boxed{f^+ : X^+ \to Z.}$

K_X-neg curves for K_{X^+}-pos. curves.

...

$$\rho(X) = \rho(X^+)$$

flips exist and termination.

Main Theorem

(X, Δ) a klt pair. $K_X + \Delta$, $\mathbb{R}$-Cartier.

$\pi : X \to U$ proj. morphism of quasi-projective varieties.

If Δ π-big, & $K_X+\Delta$ π-pseudoeffective,
or $K_X+\Delta$ π-big.

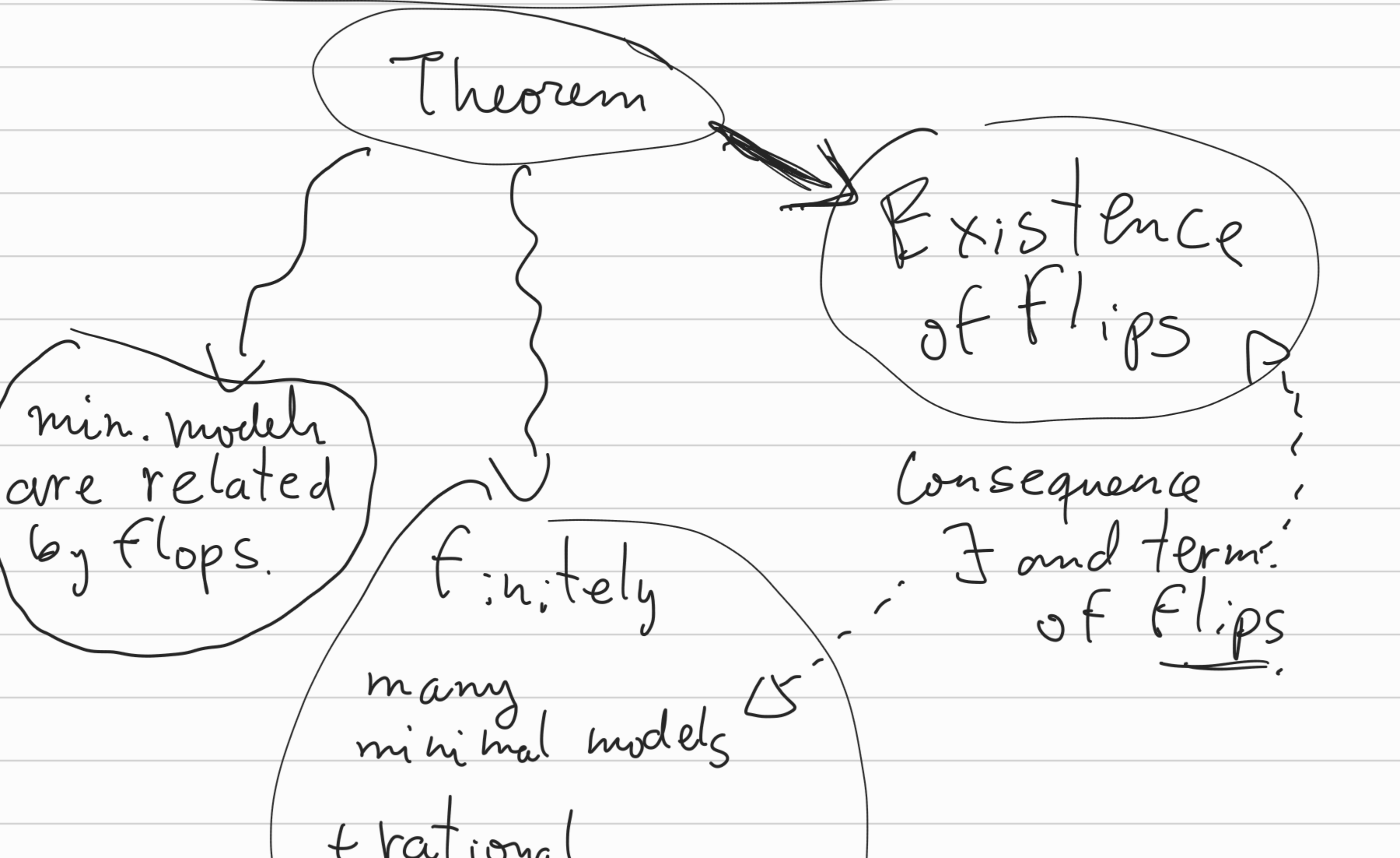
Then

a) $K_X+\Delta$ has a lt model over U.

b) $K_X+\Delta$ π-big $\Rightarrow$ $K_X+\Delta$ has a lc. model over U.

c) If $K_X+\Delta$ is $\mathbb{Q}$-Cartier.
The Cox Ring.

$$R(\pi,\Delta)=\bigoplus_{m\in\mathbb{N}}\pi_*\mathcal{O}_X(\lfloor m(K_X+\Delta)\rfloor).$$



polytope things

effective log threshold is rational

Cox rings are f.g.

Flips with scaling terminate

X Mori dream spaces

$K_X + \Delta$ not pseff it ends at Mori Fibre

Main

log terminal models we can control what is getting extracted

?

Inversion of adjunction

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l.t models are not unique.

Cor $\pi : X \to U$
$(K_X+\Delta)$ klt and Δ big over U.

Given $\phi_i : X \dashrightarrow Y_i$ l.t. models over U.

$Y_1 \overset{\text{bir}}{\dashrightarrow} Y_2$ given by a sequence of $(K_{Y_1} + \phi_{1*}\Delta)$-flops.

"Ex" 3-dim Y_1 term. sing.

and $K_{Y_1} + \Delta$ nef

flops do not change the index of K_{Y_1}

{min integer m s.t mK_Y is Cartier}.

Def $\pi : X \to U$.

V f.d. affine subspace of $WDiv_{\mathbb{R}}(X)$. Fix $A \geq 0$ $\mathbb{R}$-div.

- $V_A = \{ \Delta \mid \Delta = A + B,\ B \in V \}$

- $\mathcal{L}_A(V) = \{ \Delta = \underline{A} + \underline{B} \in V_A \mid K_X + \Delta$ is l.c., $\underline{\underline{B \geq 0}} \}$

- $\underline{\mathcal{E}_{A,\pi}(V)} = \{ \Delta \in \mathcal{L}_A(V) \mid K_X + \Delta$ is ps-eff $\}$

- $\underline{\mathcal{N}_{A,\pi}(V)} = \{ \quad // \quad$ nef $\}$

For $\phi : X \dashrightarrow Y$ bir. contraction

$\underline{\mathcal{W}_{\phi,A,\pi}(V)} = \{ \Delta \in \mathcal{E}_{A,\pi}(V) \mid \phi$ is weak log canonical model for $(X, \Delta) \}$

For $\psi : X \dashrightarrow Z$ rat. map

$A_{\psi, A, \pi}(V) = \{ \Delta \in \mathcal{E}_{A,\pi}(V) \mid$
ψ is the ample model $(X, \Delta)\}$

Cor) If $\exists\ \Delta_0 \in V$ s.t
$K_X + \Delta_0$ is klt.
and we take A to be general ample $\mathbb{Q}$-divisor with no components in common with V.

(B to always be ≥ 0)

$\Rightarrow$

1) There are finitely many bir. contractions $\phi_i : X \dashrightarrow Y_i$

$\varphi_i : X \dashrightarrow Y_i$

$1 \leq i \leq p$. s.t.

$$\boxed{\underbrace{\mathcal{E}_{A,\pi}(V)}_{K_X+\Delta}} = \boxed{\bigcup_{i=1}^{p} W_{\phi_i, A, \pi}(V)}$$

(each W_i is a rat. polytope).

If $\phi : X \dashrightarrow Y$ is l.t. model

for (X, Δ) with $\Delta \in \mathcal{E}_{A,\pi}(V)$

$\phi = \phi_i$

2) $\mathcal{E}_{A,\pi}(V)$ is decomposed

similarly by A_j, for

finitely many $\psi_j : X \dashrightarrow Z$.

3) for "i" (ϕ_i)

we get some j (ψ_j).

and a morphism

$f_{i,j} : Y_i \to Z_j$, s.t.

$W_i \subseteq \overline{A_j}$.

$\Rightarrow$ $\mathcal{E}_{A,\pi}(V)$ is a rational polytope

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Cor $\boxed{\text{Fano}}$

$\pi : X \to U$, U affine

proj. X $\mathbb{Q}$-factorial.

$K_X + \Delta$ d.l.t. $-(K_X+\Delta)$ ample.

Then X is a Mori Dream Space.

We can run D-MMP for any divisor.

Cor. (X, Δ) $\mathbb{Q}$-factorial klt.

$\pi : X \to U$.

$K_X + \Delta$ not ps-eff.

Then we can run an $(K_X+\Delta)$-MMP that ends with MFS

$g : Y \to W$

It is not proven that it can be run in any way.

(Flips exist).

We do not get termination in general.

MMP with scaling flips terminate (every sequence of flips.)

Cor) $\pi : X \to U$. (X, Δ) $\mathbb{Q}$-factorial klt.

Δ π-big..

$C \geq 0$. divisor.

If $K_X + \Delta + C$ is klt and π-nef.

We can run $(K_X + \Delta)$-MMP. with scaling of C.

$\widetilde{\text{MMP with scaling}}$.

$$K_{X_i} + \underbrace{\Delta_i + \lambda C_i}$$

$X \dashrightarrow X_i \dashrightarrow$

at each step $K_{X_i} + \Delta_i + \lambda C_i$ is nef

$\lambda_0 = 1 > \lambda_1 > \cdots$

Special log terminal models

<u>Cor</u>) (X, Δ) l.c. $f : W \to X$ log resolution

$K_X + \Delta_0$ klt.

We want $\mathcal{E}$

a) $\mathcal{E}$ contains only valuations with log disc ≤ 1 and

b) center of those with l.d $= 1$ does not contain <u>non-klt</u> centers.

$\Rightarrow$ We can find bir $\pi : Y \to X$.

exc. div. = elements of $\mathcal{E}$.

1st (X, Δ) is klt.

$\mathcal{E}$ = all ex. div. l.d ≤ 1.

$\Rightarrow \pi : Y \to X$.

we get a terminal model

2nd Take $\mathcal{E}$ = empty.

we get a log terminal model.

Fin.

3-fold sing. minimal models.

Abelian 3-fold.

$\mathbb{Z}_2 \curvearrowright A$ involution

(-1).

$A \to \boxed{X}$ 2^6 isolated sing.

is what we want.

K_X is $\mathbb{Q}$-trivial.

Index of the canonical is 2

that is a invariant under flops.

$\Rightarrow$ min. models are singular.